Discrete Mathematics Summary

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Advanced pigeonhole principle:
p pigeons, h holes, $p \geq (t-1)h$ then exists at least 1 hole with at least t pigeons.

Inclusion& exclusion theorem:
$V_1, \ldots, V_k$ finite sets, $|V_1 \cup \ldots \cup V_k| = \sum_{r=1}^{k} (-1)^{r+1} \sum_{1 \leq i_1 < \ldots < i_r \leq i_k} |V_{i_1} \cap \ldots \cap V_{i_r}|$

Or V finite, $V_1, \ldots, V_k \subseteq V$ then $|V \setminus (V_1 \cup \ldots \cup V_k| = |V| + \sum_{r=1}^{k} (-1)^r \sum_{1 \leq i_1 < \ldots < i_r \leq i_k} |V_{i_1} \cap \ldots \cap V_{i_r}|$

Alphabet A with size $|A| = r$, then if we write n symbols

- Word: $x_1, \ldots, x_n = (x_1, \ldots, x_n) \in [r]^n$ and
- shift $x_1, \ldots, x_n \to x_2, \ldots, x_n, x_1$.
- Period of word: least number of shifts that gives original word back.
- A-periodic period of word is n.
- Circular word/necklace equiv. class of shift maps, i.e. $w = v$ if v can be otained from w by some number of shifts.
- $N(n, r) = |$neclaces length, n alphabet size $r|$ and $A(n, r) = |$A-periodic necklaces length n, $|A|$
- $r^n = \#\text{all words} = \ldots = \sum_{D|n} d \cdot A(d, r)$

Möbius function: $\mu(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } e_i > 1 \text{ for some } i \\ (-1)^k \text{ if } e_1 = \ldots = e_k = 1 \end{cases} \quad \Rightarrow \sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ oth.} \end{cases}$

Möbius inversion formula:
$F, G : \mathbb{N} \to \mathbb{R}$ satisfy $F(n) = \sum_{d|n} G(d) \Rightarrow G(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$

After some equivalences: $A(n,r) = \sum_{d|n} \frac{1}{d} \left(\sum_{x|d} \mu(x) r^{d/x} \right)$

Euler's Totiënt function: $\phi(n) := \#\{1 \le i \le n : \gcd(i,n) = 1\}$.
Thm $\phi(n) = \sum_{d|n} \mu(n/d) d = \sum_{d|n} \mu(d) \frac{d}{n}$

Thm $N(n,r) = \frac{1}{n} \sum_{d|n} \phi(n/d) r^d$ & $N(n,r) = \frac{1}{n} \sum_{i=1}^{n} r^{\gcd(i,n)}$

SHIFT MAP σ s.t. $\sigma(w_1, \ldots, w_n) = w_2, \ldots, w_n, w_1$ and REVERSION MAP: τ s.t. $\tau(w_1, \ldots, w_n) = w_n, \ldots, w_1$.
Therefore we get $w \equiv_{\text{NL}} v$ iff $v = (\sigma \circ \ldots \circ \sigma)(w)$
$w \equiv_{\text{Br}} v$ iff v obtained from w by some seq. of applications of $\sigma \& \tau$.
Note:$(\tau \circ \sigma)(w) = (\sigma^{-1} \circ \tau)(w) \Rightarrow v \equiv_{\text{Br}} w$ if $v = (\sigma^{(i)} \circ \tau^{(j)})(w)$.
$B(n,r) = \#$bracelets $S(n,r) = |\mathcal{S}(n,r)| = \#\{[[w]_{\text{NL}}$ s.t. $[\tau(w)]_{\text{NL}} = [w]_{\text{nl}}\}$
Therefore $N(n,r) = 2B(n,r) - S(n,r)$

After this, there follows a lot of equivalence with $S(n,r)$ but do not think those are really important.
Cayley theorem: # trees on $[n] = n^{n-2}$

G, H graphs, then $G \cong h$ if exists bijection $\phi : V(G) \to V(H)$ s.t. $uv \in E(G) \Leftrightarrow \phi(u)\phi(v) \in E(H)$.

Theorem:
H, H' both s.t. $V(H') = V(H) = [n]$ then $\exists \pi \in S_n$ that is isomorphism $H \to H'$.
Pólya theorem: $u_n = (1 + o_n(1)) \cdot \frac{2^{\binom{n}{2}}}{n!}$ where $o_n(1) \xrightarrow{n\to\infty} 0$.

$\mathcal{C}(G) = \{H$ Graph on $[n] : H \cong G\}$ then $|\mathcal{C}(G)| = \frac{n!}{\text{aut}(G)}$ here $\text{aut}(G) = |\text{Aut}(G)| = |\phi : G \to G$ isom.$|$

$p \in S$ not id. $\Rightarrow \exists k geq 2, i_1, \ldots, i_k$ s.t. $\pi(i_1) = i_2, \ldots, \pi(i_k) = i_1$.

RECURRENCE: $a_n = f(a_{n-1}, \ldots, a_{n-m}, N)$ then $\mathcal{S} = \{(a_n)_{n\ge 0}$ solves recurrence$\}$ and $\mathcal{S}$ is lin. vec. space.
CHAR. POL $p(z) = z^m + c_1 z^{m-1} + \ldots + c_{m-1} z + c_m$ and FUND. THM OF ALG. $p(z) = (z - r_1) \cdot \ldots \cdot (z - r_m)$

Lemm: $r \neq 0$ root of $p(z)$ of mult. at least i iff $a_n = n^i r^n$ is sol. of recur.
Thm $r_1, \ldots, r_k$ roots with mult. $m_1, \ldots, m_k \Rightarrow \mathcal{S} = \text{span}(\{r_1^n, \ldots, n^{m_1-1} r_1^n, \ldots, r_k^n \ldots n^{m_k-1} r_k^n\})$

Any solution $a_n = \sum_{ij} n^i r_j^n$. Find λ_{ij} via in. cond. $\begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = M\underline{\lambda}$ with $M \in \mathbb{R}^{m \times m}$.

If roots distinct $\Rightarrow$ i'th column of M would like $\begin{pmatrix} 1 \\ r_i \\ \vdots \\ r_i^{m-1} \end{pmatrix}$ called Vandermonde matrix.

$\deg(M) = \prod_{i?j} (r_j - r_i)$

Inhom. recurrence: $a_n + c_1 a_{n-1} + \ldots + c_m a_{n-m} = f(n)$.
Cor: a_n, b_n solves inhom. recurrence $\Rightarrow$ $a_n - b_n$ solves hom. version, i.e., $f(n) = 0$.
Cor: $a_n^{(p)}$ solves inhom. recur $\Rightarrow$ all solutions of the form $a_n = a_n^{(p)} + p_1(n) r_1^n + \ldots + p_k(n) r_k^n$.
Thm: $a_n + c_1 a_{n-1} + \ldots + c_m a_{n-m} = Q(n) \cdot r^n$ then exists sol $a_n^{(p)} = R(n) \cdot r^n$ where $\deg(R) = \deg(Q) + \text{mult}(r = \text{ in } P)$ and P is char. pol of hom version.

Generalized binomial coeff. $\alpha \in \mathbb{R}, k \in \mathbb{Z}_{\geq 0} \Rightarrow \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$
$\binom{\frac{1}{2}}{n} \cdot (-1)^{n-1} \cdot (1/4)^{n-1}$ for $n \geq 1$.
$\binom{-k}{n} = \binom{n+k-1}{k-1} \cdot (-1)^n$ for all $k \in \mathbb{N}$. **gen. Newton's binom** $(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$ if $|z| < 1$.

Dyck word: $w \in \{0,1\}^n$ s.t. $\#1's = \#0's$
Dyck path: $n \times n$ grid, path from $(0,0)$ to (n,n) stays on/above same height/above diagonal.
Let C be event that A has more as B after t votes counted, then

$$\mathbb{P}(C) = \frac{\#\text{walks that stay non-negative}}{\#\text{All walks}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}$$

Catalan number: $c_n = \#\{\text{triangulations of a } n+2 \text{ gon for } n \geq 1\} = \frac{1}{n+1}\binom{2n}{n}$

$\pi \in S_n$ of $[n]$ contains the pattern $\sigma \in S_k$, if there exists $i_1 < i_2 < \ldots < i_k$ s.t. $\forall 1 \leq a, b \leq k$, we have $\pi(i_a) < \pi(i_b)$,iff $\sigma(a) < \sigma(b)$
π avoids σ if π does not contain σ.
$\text{Av}(n, \sigma) := \{\pi \in S_n : \pi \text{ avoids } \sigma\} \Rightarrow \text{av}(n, \sigma) = |\text{Av}(n, \sigma)|$
$r_n : [n]] \to [n]$ reverse map: $r_n(i) = n + 1 - i$.
Thm $\pi \in S_n$ avoids $\sigma \in S_k$ iff $r_n \circ \pi$ avoids $r_k \circ \sigma$ iff $\pi \circ r_n$ avoids $\sigma \circ r_k$.
$\text{av}(n, 231) = \text{av}(n, 312) = \text{av}(n, 213) = \text{av}(n, 132) = \frac{1}{n+1}\binom{2n}{n}$ and $\text{av}(n, 123) = \text{av}(n, 321)$

If we have a sequence, $a_0, a_1, \ldots$, then we have

- Ordinary generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$, note that $A'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n$, $\int_0^x A(x)dx = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}$
  We see that $[z^n]A(z) = a_n$, and $A^{(n)}(0) = n!a_n$.
  Convolution formula: $A \cdot B = \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right)$

  Radius of convergence of A : $\rho = \sup\{r : A(z) \text{ conv. abs.} \forall |z| < r\}$ with $A(z)$ def. on $(-\rho, \rho)$ and smooth.

  - $\rho > 0$ then $A', \int A$ have positive rad. of conv.
  - A, B positive rad. of conv $\Rightarrow A + B, A \cdot B$ positive rad. of conv.
  - $|a_n| < K^n$ for some $K \in [0, \infty) \Rightarrow \rho > 0$

  Examples:

  - $1, 1, 1, 1, \Rightarrow A(z) = 1 + z + z^2 + \ldots = \frac{1}{1-z}$ (if $|z| < 1$)
  - $1, 2, 3, \ldots \Rightarrow A(z) = 1 + 2 + z + 3z^2 = \frac{1}{(1-z)^2}$
  - $0, 1, \frac{1}{2}, \ldots \Rightarrow A(z) = z + \frac{z}{2} + \ldots = -\ln(1-z)$
  - Right shift, $zA(z)$ and left shift $\frac{A(z)-a_0}{z}$
  - A lot of examples

- Exponential generating function $\hat{A}(z) := \sum_{n \geq 0} a + n \cdot \frac{z^n}{n!}$
  $\hat{A}'(z) = \sum_{n=0}^{\infty} a_{n+1} \frac{z^n}{n!}$ i.e. left shift, $\int_0^z \hat{A}(x)dx = \sum_{n=1}^{\infty} a_{n-1} \cdot \frac{z^n}{n!}$ i.e. right shift.
  Binomial convolution formula: $\hat{A}(z) \cdot \hat{B}(z) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right)$

Multinomial coef: $\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \ldots n_k!}$ if $n_1 + \ldots + n_k = n$, oth. 0

$o_{n,k} := \#\{\text{ordered part. of } [n] \text{ into precisely } k \text{ non-empty sets}\} = \sum_{j=0}^{n} \binom{n}{j} o_{n-j,k-1} \cdot o_{j,1}$

$\hat{O}_k(z) := \sum_{n=0}^{\infty} o_{n,k} \frac{z^n}{n!} = \ldots = \sum_{n=)}^{\infty} \frac{z^n}{n!} \cdot \left(\sum_{j=0}^{k} \binom{k}{j} j^n (-1)^{k-j} \right)$ so new formula for $o_{n,k}$

$u_{n,k} = \frac{1}{k!} \cdot o_{n,k} = \frac{1}{k!} \cdot \sum_{j=0}^{k} \binom{k}{j} j^n (-1)^{k-j} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ Stirling number of 2nd kind

$\hat{U}_k(z) = \frac{1}{k!}(e^z - 1)^k$ and Bell number $b_n := \sum_{k=1}^{n} u_{n,k} = \sum_{k=0}^{\infty} u_{n,k}$. Therefore $\hat{B}(z) = e^{e^z-1}$

Thm $b_n = \frac{1}{e} \cdot \sum_{k=0}^{\infty} \frac{k^n}{k!}$

Recursion Bernoulli: $b_0 = 1, \sum_{j=0}^{k} \binom{k+1}{j} \cdot b_j = 0$, for $k \geq 1$.

Thm: $s_{n,k} = \frac{1}{k+1} \cdot \sum_{i=0}^{k} \binom{k+1}{i} \cdot b_i \cdot n^{k+1-i}$

We see that $\hat{S}_n(z) = \hat{B}(z) \cdot \left(\frac{e^{nz}-1}{z}\right)$ and $\hat{S}_n(z) = \sum_{k=0}^{\infty} s_{n,k} \frac{z^k}{k!}$. If we work this out, we get $s_{n,k} = \frac{1}{k+1} \cdot \sum_{l=0}^{k} \binom{k+1}{l} b_l n^{k+1-l}$i.e., bernoulli's formula

$\left[\begin{matrix} n \\ k \end{matrix} \right] = s_k = \#\pi \in S_n$ with exactly k cycles Stirling number of first kind

$(z)_n = z(z-1)\ldots(z-n+1) \Rightarrow (z_n) = \sum_{k=0}^{n} (-1)^{n-k} \left[\begin{matrix} n \\ k \end{matrix} \right] z^k$ since $\left[\begin{matrix} n+1 \\ k \end{matrix} \right] = \left[\begin{matrix} n \\ k-1 \end{matrix} \right] + n \left[\begin{matrix} n \\ k \end{matrix} \right]$

Note:$\left[\begin{matrix} n \\ k \end{matrix} \right] = \sum_{1 \leq i_1 < \ldots < i_{n-k} \leq n} i_1 \ldots i_{n-k}$

Stirling inversion $(a_n)_n, (b_n)_n$, with $b_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} a_k, \forall n \Leftrightarrow a_n = \sum_{k=0}^{n} (= 1)^{n-k} \left[\begin{matrix} n \\ k \end{matrix} \right] b_k$

t_n tree on $[n]$, r_n rooted tree $(T, \rho) \in V(T)$, we see that $r_n = n \cdot t_n$. Let $r_{k,n} = \#$ rooted trees with $\deg(\rho) = k$.
After a lot of computations, we get that $r_i = i^{i-1}$

Ramsey number $R(s,t) = \min\{n \in \forall \text{red,blue colouring of } E(R_n), \exists a \text{ red } K_t \text{ or blue } K_s\}$
Thm $R(s,t) < \infty, \forall s, t.$ and **Prob meth.** $(\sqrt{2})^t \leq R(t,t) \leq 4^t$
Erd. thm $\binom{n}{t} 2^{1-\binom{t}{2}} < 1 \Rightarrow R(t,t) > n$.cor $R(t,t) \geq \sqrt{2}^t$
Erd-szek thm $R(s+1, t+1) \leq R(s, t+1) + R(s+1, t)$.Cor $R(s,t) \leq 2^{s+t}$

$\chi(G)$= Chromatic number of G, smallest k for which a k colouring exists.
$A \subseteq V(G)$ is stable set if $ab \notin E(G)$, for all $a, b \in A$.

STABILITIY NUMBER/INDEPENDENCE NUMBER OF G, $\alpha(G) = |\text{largest stable set}|$. Therefore $\chi(G) \geq \frac{v(G)}{\alpha(G)}$
GIRTH (G) := length of shortest cycle in G.
Thm $\forall k, l, \exists G$ with $\chi(G) > k$ and $\text{girth}(G) > 1$

$a_n = O(b_n) \Rightarrow \exists c; a_n \leq cb_n$ and $a_n = \Omega(b_n) \Rightarrow \exists c : a_n \geq cb_n$
$a_n = o(b_n) \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} \to 0$ and $a_n = \omega(b_n) \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} \to \infty$

CROSSING NUMBER $\text{cr}(G)$ = least number of corssings in a drawing of G. $\text{cr}(G) = 0 \Leftrightarrow G$ planar.
Conj. $\text{cr}(K_{n,m}) = \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor \cdot \dots \cdot \lfloor \frac{m-1}{2} \rfloor$.
Euler's formula: $2 = v(g) - e(g) + f(G)$ if G conn and planar.
Lem: G Planar $\Rightarrow e(G) \leq 3v(G)$. If G planar and $v(G) \geq 3 \Rightarrow e(G) \leq 3v(G) - 6$
Cors. num. ineq. $\text{cr}(G) \geq \frac{e(G)^3}{64v(G)^2}$ prov. $e(G) \geq 4v(G)$.Cor: $\text{cr}(K_n) \geq \frac{\binom{n}{2}^3}{62n^2} = \Omega(n^4)$
$\text{rcr}(G)$ = min number of crossings in drawiang with all edges line segments.
Thm $\text{cr}(G) = 0 \Leftrightarrow \text{rcr}(G) = 0$.
B& D thm 1) $\text{cr}(G) \leq 3 \Rightarrow \text{rcr}(G) = \text{cr}(G)$ 2) $\forall k, \exists G$ s.t. $\text{cr}(G) = 4, \text{rcr}(G) > k$.

$I(P, L) = |\{(p, l), p \in P, l \in L\}$ so number of pairs (point,line) s.t. point on line.
$u(n) = \max_{\substack{P \subseteq \mathbb{R}^2 \\ |P|=n}} \{\{p, q\} \in \binom{P}{2} : |p - q| = 1\}$
Thm $I(P, L) \leq 4(|P||L|)^{2/3} + |L| + 4|P|$,,**Thm** $\exists c : u(n) > n^{1+\frac{c}{\ln\ln(n)}}$ **thm** $u(n) = O(n^{4/3})$

INTERSECTING FAMILY $F \subseteq 2^V$ with V finite,s.t. $A \cap B \neq \emptyset, \forall A, B \in F$.
Is $k-$ uniform, if $|A| = k, \forall A \in F$.
$F_v := \{A \in \binom{V}{k} : v \in A\}$.
Thm: $n := |V|, k \leq n/2, F \subseteq \binom{V}{k}$ intersecting, then $|F| \leq \binom{n-1}{k-1}$

$$u_{n,k} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{o_{n,k}}{k!} \Leftarrow \# \text{ unordered partition with } k \text{ parts on } [n] \Leftarrow \text{Stirl. second kind}$$

$$\hat{U}_k(z) = \sum_{n=0}^{\infty} u_{n,k} \frac{z^n}{n!} = \frac{1}{k!}(e^z - 1)^k$$

$$o_{n,k} = \sum_{j=0}^{k} \binom{k}{j} j^n (-1)^{k-j}$$

$$b_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{n!} \Leftarrow \text{Bell number}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{1 \le i_1 < \ldots < i_{n-k} \le n} i_1 \ldots i_{n-k} \Leftarrow \{\# \pi \in S_n \text{ with exactly k cycles}\} \Leftarrow \text{Stir. first kind}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n \begin{bmatrix} n \\ k \end{bmatrix}$$

$$S_{n,k} = \frac{1}{k+1} \sum_{l=0}^{k} b_l n^{k+1-l}$$

$$\hat{S}_k(z) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!} = \ln \left(\frac{1}{1+z} \right)^k$$

$$b_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \Leftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} b_k$$

$$A(z) \cdot B(z) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-1} \right) \& \hat{A}(z) \cdot \hat{B}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \right)$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} b_k$$